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Algorithms & Data Structures Homework 1 HS 18

Exercise Class (Room & TA): ____________________

Submitted by: ____________________

Peer Feedback by: ____________________

Points: _____

Exercise 1.1 *Induction.*

1. Prove via mathematical induction, that the following holds for any positive integer n:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} .$$

2. Prove via mathematical induction that for any positive integer n,

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i .$$

Exercise 1.2 *Acyclic Graphs.*

Definitions:

- A graph is **acyclic** if there are no cycles. A **cycle** is a nontrivial path from vertex a to itself.
- A graph is **connected** if there is a path between every pair of vertices.
- An acyclic graph is called **non-trivial** if it has at least one edge.

For a given connected acyclic graph $G = (V, E)$, avoid using induction and prove the following:

1. There is a unique path between any pair of vertices u and v, such that $u \neq v$.
2. Adding an edge between any pair of vertices creates a cycle.
3. Show that any non-trivial acyclic graph has at least two vertices of degree 1.
   Hint: *consider some longest path.*

Exercise 1.3 *Number of Edges in Graphs* **(1 point)**.

Definition:

- A **Tree** is an acyclic graph that is connected.

1. Show by mathematical induction that the number of edges in a tree with n vertices is $n-1$.
2. Prove or disprove that every graph with n vertices and $n-1$ edges is a tree.

Exercise 1.4 *Bipartite Graphs* **(2 points)**.

1. Consider the following lemma: *If G is a bipartite graph and the bipartition of G is X and Y, then*

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v) \tag{1}$$

   Then, use the lemma to prove that you can not cover the area in Figure 1, with the given tiles of size 1×2 and 2×1, depicted in the same figure.

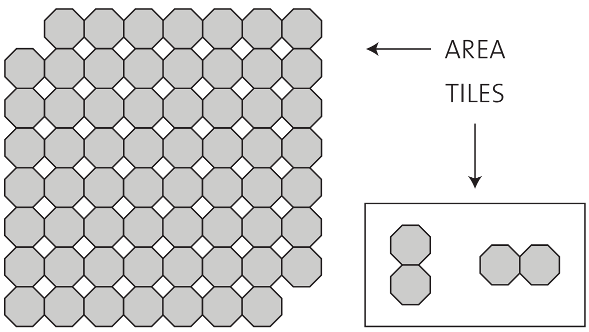

Figure 1: Cover the area with the given tiles

2. Coloring Bipartite Graphs

   Suppose you are given a map with n vertical lines. The areas of the map (ie the areas between the lines) must be colored such that any two neighboring areas have different colors. Prove by mathematical induction that any such map can be colored with exactly two colors. Hint: Suppose you start with a map with two vertical lines, dividing the map into three regions, colored red, blue, and red from right to left. What happens if you draw a vertical line through the blue region? How can you modify the colors of the regions to maintain the property that neighboring regions have different colors?

Exercise 1.5 *Sudoku.*

The classic Sudoku game involves a 9×9 grid. This grid is divided into nine 3×3 nonoverlapping subgrids, called blocks. The grid is partially filled by digits from 1 to 9. The objective is to fill this grid with digits so that each column, each row, and each block contains all of the digits from 1 to 9. Each digit can only appear once in a row, column or block (see Figure 2).

8			1	5		6		
			3				4	1
5						7		
					9		6	2
				3				
1	4		8					
		8						9
2	9				1			
		5		9	7			6

(a) A Sudoku puzzle

8	7	4	1	5	2	6	9	3
6	2	9	3	7	8	5	4	1
5	3	1	9	6	4	7	2	8
3	5	7	4	1	9	8	6	2
9	8	2	7	3	6	1	5	4
1	4	6	8	2	5	9	3	7
7	6	8	5	4	3	2	1	9
2	9	3	6	8	1	4	7	5
4	1	5	2	9	7	3	8	6

(b) Solution

Figure 2: Sudoku

Model this as a graph problem: give a precise definition of the graph involved and state the specific question about this graph that needs to be answered. What is the maximum vertex degree of this graph?

Submission: On Monday, 1.10.2018, hand in your solution to your TA *before* the exercise class starts.